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Asymptotic High-Frequency Modes of Homogeneous Waveguide Structures with Impedance Boundaries

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Abstract—**Homogeneous waveguides with both isotropic and anisotropic impedance boundaries are considered and asymptotic high-frequency mode properties are systematically derived. Among the new results are orthogonality properties of the asymptotic HF fields, existence of self-dual solutions, construction of stationary functionals, and an explicit formula for the calculation of the asymptotic attenuation coefficient for the general waveguide.**

I. Introduction

THE PROBLEM of guided waves in structures of large transverse dimensions has many applications, e.g., in antenna feed systems and millimeter and submillimeter wave engineering. A paper on HE_{11} modes in large waveguides was recently published [1]. The theory presented was, however, mainly limited to special geometries and loaded with unnecessary assumptions, which has prompted this author to attempt of a more systematic theory of asymptotic modes.

In the present study, the fields are derived through Hertzian potentials as in conventional waveguide mode analysis [2]. These potentials are expanded in asymptotic series with respect to the inverse powers of the wavenumber k and equations for the coefficients are obtained. The basic is the Helmholtz equation for the two scalar potential functions and the eigenvalue is related to the difference of the propagation factor and the free-space k value. The basic problem is independent of the true impedance properties of the boundary, as was demonstrated in [1]. Properties of the basic solutions are considered, the mode fields satisfy certain orthogonality conditions. Also, the modes are all degenerate. Two functionals are presented that are stationary for the solutions of the basic problem and give the eigenvalue as the stationary value. The eigenvalues are seen to be real so that no attenuation is connected with the basic problem.

Defining two dual transformations, we see that a dual transformation of a basic solution is also a basic solution of the asymptotic waveguide problem. The most natural way, of defining and classifying the mode fields seems to be in terms of two self-dual solutions of the basic problem, because they both satisfy uncoupled boundary conditions. The problem is, then, formulated in terms of one scalar potential function only. A stationary functional is also presented for the self-dual modes. These modes are circularly polarized everywhere, whence there is no need to consider any special coordinate system in the transverse plane. As an application, the circular cylindrical geometry is analyzed for self-dual modes.

The attenuation is obtained in the next problem, which applies the solution of the basic problem. An explicit formula for the calculation of the attenuation coefficient is given. This involves the boundary conditions and is considered separately for an isotropic and an anisotropic boundary. As an example, corrugated surface is analyzed and it is found that the attenuation is decreased if the longitudinal impedance is decreased and the transverse impedance increased, which was demonstrated for special geometries in [1].

II. The Waveguide Problem

The waveguide structure considered here (Fig. 1) is uniform in the z-coordinate and bounded with any closed curve C in the transverse plane. Because of the transla-

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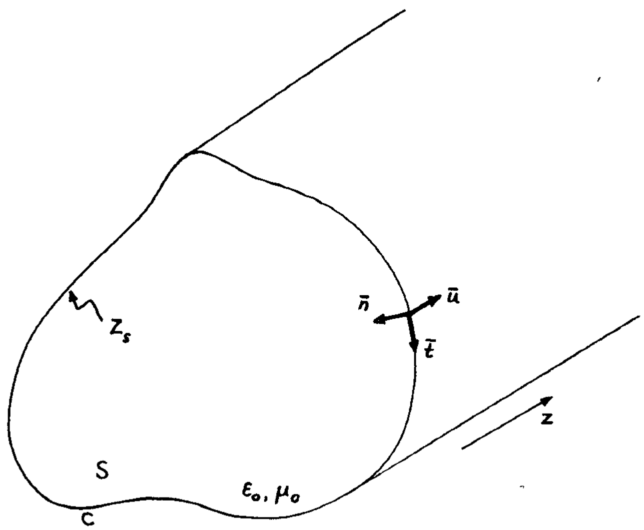

Fig. 1. The general waveguide with surface impedance Z_s.

tional invariance in z, all eigensolutions contain the z-dependence in the form $e^{-j\beta z}$. The medium inside the curve C is assumed homogeneous and the boundary conditions on C are the Leontovich impedance conditions with either isotropic or anisotropic surface impedance. In the latter case, $\boldsymbol{Z}_s$ is a dyadic. The guide may have other boundaries inside C, i.e., be multiply connected, and C may move to the infinity, whence with $Z_s=\eta$, the free-space impedance, the radiation condition is satisfied and open guides can be included in the analysis, e.g., the Sommerfeld–Goubau line. The main approximation made here is that the surface impedance is assumed independent of the frequency. If $Z_s(\omega)$ can be written as an asymptotic series in the inverse powers of k, this presents no problem in the asymptotic approach, but for nonanalytic functions (like the square root for good conductor surfaces) the case is not so simple. We postpone this argument to be treated in a further study and concentrate here on the basic theory.

A. The Hertzian Potentials

It is well known that the electromagnetic field in a straight guide can be represented in terms of two scalar functions, the Hertzian potentials $\pi(\rho)$, $m(\rho)$ [2]:

$$\boldsymbol{E}(\boldsymbol{r})=\left[\boldsymbol{u}k_c^2\pi(\boldsymbol{\rho})-j\beta\nabla\pi(\boldsymbol{\rho})+jk\boldsymbol{u}\times\nabla m(\boldsymbol{\rho})\right]e^{-j\beta z} \quad (1)$$

$$\eta\boldsymbol{H}(\boldsymbol{r})=\left[\boldsymbol{u}k_c^2 m(\boldsymbol{\rho})-j\beta\nabla m(\boldsymbol{\rho})-jk\boldsymbol{u}\times\nabla\pi(\boldsymbol{\rho})\right]e^{-j\beta z}. \quad (2)$$

Here, $\boldsymbol{u}=\boldsymbol{u}_z$ is the axial unit vector, $\boldsymbol{\rho}$ is the position vector in the transverse plane and $k_c^2=k^2-\beta^2$. The potential functions satisfy the two-dimensional Helmholtz equations

$$(\nabla^2+k_c^2)\begin{pmatrix}\pi\\ m\end{pmatrix}=0, \qquad \text{on } S. \quad (3)$$

On the boundary curve C, the following scalar impedance condition is assumed:

$$\boldsymbol{n}\times\boldsymbol{E}=-Z_s\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{H}. \quad (4)$$

The anisotropic boundary will be treated in Section V. Inserting (1) and (2) in (4) gives us the following boundary conditions for the potentials. Here, $\boldsymbol{n}$ and $\boldsymbol{t}$ are unit vectors normal and tangential to C

$$\begin{pmatrix} k\boldsymbol{n}\cdot\nabla & -\beta\boldsymbol{t}\cdot\nabla \\ \beta\boldsymbol{t}\cdot\nabla & k\boldsymbol{n}\cdot\nabla\end{pmatrix}\begin{pmatrix}\pi\\ m\end{pmatrix}=k_c^2\begin{pmatrix}-j\eta/Z_s & 0\\ 0 & Z_s/j\eta\end{pmatrix}\begin{pmatrix}\pi\\ m\end{pmatrix}. \quad (5)$$

B. The Asymptotic Series Approach

We are interested in the guided modes as the frequency is increased without limit, $\omega\to\infty$. To perform the asymptotic analysis we write every quantity dependent on the frequency as a power series in the inverse powers of k, which closely resembles the geometrical optics series method [3]. The expansions are

$$\pi=\pi_1/k+\pi_2/k^2+\cdots \quad (6)$$

$$m=m_1/k+m_2/k^2+\cdots \quad (7)$$

$$\beta=\beta_{-1}k+\beta_0+\beta_1/k+\cdots \quad (8)$$

$$k_c^2=-(\beta_{-1}^2-1)k^2-2\beta_{-1}\beta_0 k -(\beta_0^2+2\beta_{-1}\beta_1)-2(\beta_0\beta_1+\beta_{-1}\beta_2)/k-\cdots. \quad (9)$$

The terms π_n, m_n with $n<1$ must be zero, because, as we see from (1), (2), otherwise the fields would grow with increasing frequency without limit and this is excluded by normalization requirements. Also, the terms β_n with $n<-1$ are zero for the same reason. The series are asymptotic, i.e., they may not converge as $n\to\infty$, but for a sufficient large k a few terms will give a good approximation. The unknown coefficients are solved from the equations which are obtained by substituting (6) $\cdots$ (9) in (3), (5) and equating the coefficients of every power of k. In this manner, we end up with a system of equations with rising power n.

C. $n=-1$ Equations

The Helmholtz equations (3) imply

$$(\beta_{-1}^2-1)\begin{pmatrix}\pi_1\\ m_1\end{pmatrix}=0 \quad (10)$$

which gives, assuming the Hertzian potential coefficients nonzero,

$$\beta_{-1}=\pm 1 \quad (11)$$

or the asymptotic value of β is k as the frequency is increased. It is well understood that the waves in oversized waveguides travel much like waves in free space. Here we limit the solution to waves going in the positive z-direction, whence only the + sign in (11) is approved.

D. $n=0$ Equations

From (3) we have now

$$\beta_0=0 \quad (12)$$

or there is no first-order correction to the value k of the propagation factor. The series (8) and (9) can be simplified to read

$$\beta=k+\beta_1/k+\beta_2/k^2+\cdots \quad (13)$$

and

$$k_c^2=-2\beta_1-2\beta_2/k-\beta_1^2/k^2-\cdots. \tag{14}$$

More equations of the order $n=0$ are obtained from the impedance boundary conditions (5)

$$\left.\begin{aligned}\boldsymbol{n}\cdot\nabla\pi_1-\boldsymbol{t}\cdot\nabla m_1=0\\ \boldsymbol{n}\cdot\nabla m_1+\boldsymbol{t}\cdot\nabla\pi_1=0\end{aligned}\right\}, \quad \text{or} \quad \nabla\pi_1-\boldsymbol{u}\times\nabla m_1=0. \tag{15}$$

Denoting

$$\boldsymbol{\phi}(\boldsymbol{\rho})=\nabla\pi(\boldsymbol{\rho})-\boldsymbol{u}\times\nabla m(\boldsymbol{\rho}) \tag{16}$$

we thus have $\boldsymbol{\phi}_1=0$ at the boundary curve C. It is remarkable that this condition does not involve the surface impedance Z_s if it is finite and nonzero. Hence, the basic potential coefficients π_1, m_1 do not depend on the surface impedance, nor does β_1, the basic correction coefficient of the propagation factor.

Also, (15) shows us that in general, both π_1 and m_1 are nonzero, because they are coupled in the boundary conditions. Thus, TE^z and TM^z waves are not in general possible. TE^z waves are only possible if m_1 satisfies two boundary conditions: $\nabla m_1=0$ on C or $\partial m_1/\partial n=0$ and $\partial m_1/\partial t=0$ at the same time. This is satisfied only for some special geometries.

E. $n=1$ Equations

From the Helmholtz equations (3), the order 1 equations for the Hertzian potentials can be written

$$\left(\nabla^2-2\beta_1\right)\begin{pmatrix}\pi_1\\ m_1\end{pmatrix}=0. \tag{17}$$

This is a pair of Helmholtz equations for the potentials and the boundary conditions are (15). Denoting

$$h^2=-2\beta_1 \tag{18}$$

we have the standard form $(\nabla^2+h^2)\pi_1=0$ and $(\nabla^2+h^2)m_1=0$. The solution depends only on the boundary curve C (or curves C_i if more than one are present), and nothing else. Evidently, there exists an infinity of eigenvalues and corresponding eigenvectors like for the well-known Dirichlet and Neumann eigenvalue problems. In fact, this can be deduced from the stationary functional to be given in the next section, applying the method described in [4].

More equations of the order $n=1$ are obtained from the boundary conditions. Since they involve the relations of π_2, m_2, on π_1, m_1, they are postponed to a later section and the basic potentials are considered first.

III. The Basic Asymptotic Field

From (1) and (2) we may write expressions for the basic asymptotic approximation of the electromagnetic field

$$\begin{aligned}\boldsymbol{E}_0(\boldsymbol{r})&=-j(\nabla\pi_1-\boldsymbol{u}\times\nabla m_1)\exp\left(-jkz-j\beta_1 z/k\right)\\ &=-j\boldsymbol{\phi}_1\exp\left(-jkz-j\beta_1 z/k\right)\end{aligned} \tag{19}$$

$$\eta\boldsymbol{H}_0(\boldsymbol{r})=-j\boldsymbol{u}\times\boldsymbol{\phi}_1\exp\left(-jkz-j\beta_1 z/k\right). \tag{20}$$

These equations first show us the following interdependence of the fields:

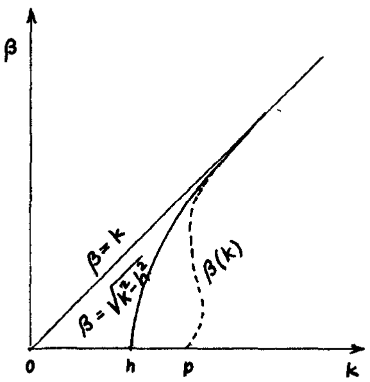

Fig. 2. Approximations of the dispersion curve $\beta=\beta(k)$.

$$\begin{aligned}\boldsymbol{E}_0&=-\eta\boldsymbol{u}\times\boldsymbol{H}_0\\ \boldsymbol{H}_0&=\frac{1}{\eta}\boldsymbol{u}\times\boldsymbol{E}_0.\end{aligned} \tag{21}$$

These are the well-known relations of the plane-wave fields. That is, the fields in the asymptotic high-frequency mode are TEM and orthogonal to each other, and the proportion of the magnitudes is η. Secondly, from the boundary conditions (15), the basic asymptotic fields are seen to satisfy

$$\boldsymbol{E}_0=0 \qquad \boldsymbol{H}_0=0, \qquad \text{on } C. \tag{22}$$

This condition was obtained in [1] through a local plane-wave consideration, which is unnecessary.

Thirdly, the propagation constant equals k up to the second order term

$$\beta\simeq k(1-h^2/2k^2)=k+\beta_1/k. \tag{23}$$

If k is large enough, (23) is an approximation of $\beta=\sqrt{(k^2-h^2)}$, or h can be interpreted as the equivalent cutoff wavenumber for the asymptotic mode. The dispersion curve $\beta=\beta(k)$ is approximated by the hyperbola defined by h more closely than by the line $\beta=k$. (See Fig. 2.) To obtain more complete view of the function $\beta(k)$, one should try to solve for the real cutoff wavenumber $k=p$ and possibly expand the fields as an asymptotic series around it and then try to match the two curves. The low-frequency properties, however, are dependent on the surface impedance Z_s, whereas (23) is not.

A. Orthogonality of the Basic Modes

We are able to show that the solutions of (17) plus (15) satisfy certain orthogonality properties. To be specific, it will be proven that the mode vectors $\boldsymbol{\phi}_1^i$ corresponding to the eigenvalues h^i, $i=1,2,3\cdots$ satisfy

$$\int\boldsymbol{\phi}_1^i\cdot\boldsymbol{\phi}_1^j\,dS=0, \qquad \text{for } h^i\neq\pm h^j, \qquad \beta_1^i\neq\beta_1^j \tag{24}$$

$$\int\boldsymbol{\phi}_1^{i*}\cdot\boldsymbol{\phi}_1^j\,dS=0, \qquad \text{for } h^i\neq\pm h^j. \tag{25}$$

The domain of integration is the cross-sectional surface S, bounded by the boundary curve C (or curves C_n). In fact, substituting (16) in the integral (24) and invoking (15) we have after an application of the Gauss' law:

$$\int \boldsymbol{\phi}_1^i \cdot \boldsymbol{\phi}_1^j \, dS = \int \nabla \cdot \left(\pi_1^i \boldsymbol{\phi}_1^j + m_1^i \mathbf{u} \times \boldsymbol{\phi}_1^j \right) dS$$

$$- \int \left(\pi_1^i \nabla \cdot \boldsymbol{\phi}_1^j - m_j^i \mathbf{u} \cdot \nabla \times \boldsymbol{\phi}_1^j \right) dS$$

$$= - \int \left(\pi_1^i \nabla^2 \pi_1^j + m_1^i \nabla^2 m_1^j \right) dS$$

$$= (h^j)^2 \int \left(\pi_1^i \pi_1^j + m_1^i m_1^j \right) dS. \quad (26)$$

The first term is symmetric in i and j whereas the last one is not of a symmetric form. Because it must be symmetric, we have

$$\left[(h^i)^2 - (h^j)^2 \right] \int \left(\pi_1^i \pi_1^j + m_1^i m_1^j \right) dS = 0. \quad (27)$$

If $h^i \neq \pm h^j$, the integral in (27) must vanish and, hence, (24) is proven. The second orthogonality property (25) can be derived as above, but with the i indexed terms conjugated. Instead of (26), (27) we have

$$\int \boldsymbol{\phi}_1^{i*} \cdot \boldsymbol{\phi}_1^j \, dS = (h^j)^2 \int \left(\pi_1^{i*} \pi_1^j + m_1^{i*} m_1^j \right) dS \quad (28)$$

$$\left[(h^{i*})^2 - (h^j)^2 \right] \int \left(\pi_1^{i*} \pi_1^j + m_1^{i*} m_1^j \right) dS = 0. \quad (29)$$

Setting $j = i$ in (28) shows us that $(h^i)^2$ is real and positive, or h^i is a real quantity. Hence, from (29), the orthogonality property (25) results.

Because β_1 is a real quantity, it does not involve any attenuation of the basic mode. The earliest term with attenuation is β_2, or the attenuation constant is asymptotic to $1/k^2$, a fact which was shown to be true for corrugated waveguides in [1].

The mode vectors $\boldsymbol{\phi}_1^i$ can be taken to be real. In fact, if the potentials π_1^i, m_1^i are complex, from (15) and (17) we see that both the real and the imaginary parts of the potentials are solutions to the same eigenvalue problem with the same real eigenvalue β_1^i, whence from (16) we see that both real and imaginary parts of $\boldsymbol{\phi}_1^i$ are solutions and can be normalized real.

B. Functionals for the Eigenvalues

It is not difficult to demontrate that the following functional $J(f, g)$ is stationary for the solutions of (17) with boundary conditions (15), and gives $J(\pi_1^i, m_1^i) = (h^i)^2$:

$$J(f, g) = \frac{\int (\nabla f - \boldsymbol{u} \times \nabla g)^2 \, dS}{\int (f^2 + g^2) \, dS}. \quad (30)$$

In fact, if the first variation of (30) is equated to zero, we have writing $\boldsymbol{\theta} = \nabla f - \boldsymbol{u} \times \nabla g$

$$\int_C \left[\delta f \boldsymbol{n} \cdot \boldsymbol{\theta} + \delta g (\boldsymbol{n} \cdot \boldsymbol{u} \times \boldsymbol{\theta}) \right] dC$$

$$- \int_S \left[\delta f (\nabla \cdot \boldsymbol{\theta} + J f) + \delta g (\nabla \cdot \boldsymbol{u} \times \boldsymbol{\theta} + J g) \right] dS = 0. \quad (31)$$

Because this must be valid for any variations $\delta f, \delta g$, the integrals must vanish independently. Choosing $\delta g = 0$ and $\delta f = \alpha (\nabla \cdot \boldsymbol{\theta} + J f)^*$, we end up in the equation $\nabla^2 f + J f = 0$ on S. Likewise, we obtain the same equation for g. On the boundary C the condition becomes $\boldsymbol{n} \cdot \boldsymbol{\theta} = 0$ an $\boldsymbol{n} \cdot \boldsymbol{u} \times \boldsymbol{\theta} = 0$ or together $\boldsymbol{\theta} = 0$. Thus, f and g are solutions of (17), (15), and J is the eigenvalue $-2\beta_1$. It must be noted that the correct boundary conditions appear as natural boundary conditions of the functional, whence the class of admissible functions is the largest possible one.

With an effort comparable to the previous one, the stationarity of the following functional can also be demonstrated:

$$J(f, g) = \frac{\int |\nabla f - \boldsymbol{u} \times \nabla g|^2 \, dS}{\int \left(|f|^2 + |g|^2 \right) dS}. \quad (32)$$

The two functionals (30), (32) are equivalent for real test functions. For some complex functions, however, (30) might fall in the indeterminate form 0/0, which is not the case for (32).

IV. Duality Transformations

The asymptotic modal fields satisfy certain interesting properties with respect to duality transformations. Here we restrict ourselves to the two duality transformations satisfying the following three requirements:

1) Maxwell's equations must remain invariant;
2) the free space (intrinsic impedance η) transforms to itself;
3) the involutory property is satisfied: the dual of the dual field is the original field.

The two transformations satisfying these conditions are the right-hand and left-hand duality transforms defined by

$$\boldsymbol{E}^d = \pm j \eta \boldsymbol{H} \qquad \boldsymbol{H}^d = \mp j \boldsymbol{E} / \eta. \quad (33)$$

From here on, the double sign refers to these two transformations, the upper one to the left-hand and the lower one to the right-hand transformation. The names arise from the self-dual polarizations, as will be elucidated in Section IV-A.

Applying the transformations to the asymptotic mode fields (19), (20), leaves us with

$$\boldsymbol{E}_0^d = \pm j \boldsymbol{u} \times \boldsymbol{E}_0 \quad (34)$$

$$\boldsymbol{H}_0^d = \pm j \boldsymbol{u} \times \boldsymbol{H}_0. \quad (35)$$

That is, the dual fields are the original fields rotated 90° around the $\boldsymbol{u}$-axis and phase-shifted by 90°. Writing the fields in terms of the Hertzian potentials gives us the corresponding transformations for the potentials

$$\pi_1^d = \pm j m_1 \qquad m_1^d = \mp j \pi_1. \quad (36)$$

From (15), (17) it is seen that the dual potentials satisfy exactly the same equations as the original potentials, whence we may state that the dual fields are also asymptotic solutions to the same waveguide problem with the same eigenvalue as the original field. If the original solution is a real field $\boldsymbol{E}_0, \boldsymbol{H}_0$ with perpendicular electric and magnetic fields, the dual solutions are both multiples of another real field $\boldsymbol{E}_0', \boldsymbol{H}_0'$, in whose field pattern the electric and magnetic field lines appear interchanged. Thus, every eigenvalue of the problem (15), (17) is degenerate.

A. The Self-Dual Solutions

Applying the duality transformation, we are able to construct solutions to the eigenvalue problem that do not prefer any coordinate system in the transverse plane. These are the self-dual solutions corresponding to the two duality transformations. Self-dual fields are fields that remain invariant in the transformations, or

$$\boldsymbol{E}_0^{\pm} = \tfrac{1}{2}(\boldsymbol{E}_0 \pm j\boldsymbol{u}\times\boldsymbol{E}_0) \tag{37}$$

$$\boldsymbol{H}_0^{\pm} = \tfrac{1}{2}(\boldsymbol{H}_0 \pm j\boldsymbol{u}\times\boldsymbol{H}_0). \tag{38}$$

These are circularly polarized fields, i.e., they satisfy $\boldsymbol{E}_0^{\pm}\cdot\boldsymbol{E}_0^{\pm}=0, \boldsymbol{H}_0^{\pm}\cdot\boldsymbol{H}_0^{\pm}=0$, and were known to exist in circular cylindrical guides, [5]. It is not difficult to see that the field $\boldsymbol{E}_0^+, \boldsymbol{H}_0^+$ is circularly polarized in the left-hand sense and $\boldsymbol{E}_0^-, \boldsymbol{H}_0^-$ in the right-hand sense with respect to the direction of propagation $\boldsymbol{u}$.

The existence of these self-dual solutions for the asymptotic waveguide problem can be applied to simplify the Hertzian potential problem. In fact, instead of looking for π_1, m_1, we may search for π_1^+ or π_1^- defined by

$$\pi_1^{\pm} = \tfrac{1}{2}(\pi_1 \pm jm_1)$$
$$m_1^{+} = \tfrac{1}{2}(m_1 \mp j\pi_1) = \mp j\pi_1^{\pm}. \tag{39}$$

The self-dual potentials are solutions to the following equations:

$$(\nabla^2 + h^2)\pi_1^{\pm} = 0, \qquad \text{on } S \tag{40}$$

$$\boldsymbol{n}\cdot\nabla\pi_1^{\pm} \pm j\boldsymbol{t}\cdot\nabla\pi_1^{\pm} = 0, \qquad \text{on } C. \tag{41}$$

It is important to note that in the boundary condition (41) the two potentials are not coupled as in (15). So, the self-dual problem is reduced to solving one scalar function only.

From (40), (41) it can be seen that the two self-dual solutions are simply related. In fact, we could define

$$\pi_1^{\pm} = (\pi_1^{\mp})^* \tag{42}$$

or they are complex conjugates of each other. So, there is no need to calculate both potentials.

The functional (32) can be written for one scalar function only, giving the stationary value for the solutions of (40), (41)

$$J^{\pm}(f) = \frac{\int(|\nabla f|^2 \pm j\boldsymbol{u}\cdot\nabla f^*\times\nabla f)\, dS}{\int |f|^2\, dS}. \tag{43}$$

These two functionals give the same stationary value h^2 for the two signs corresponding to the potentials $f=\pi^{\pm}$.

B. Circular Cylindrical Waveguide

As a simple application of the theory we consider the circular cylindrical structure with radius a much larger than the wavelength (Fig. 3). Looking for self-dual solutions, we start from (40), (41). A finite solution of (40) is

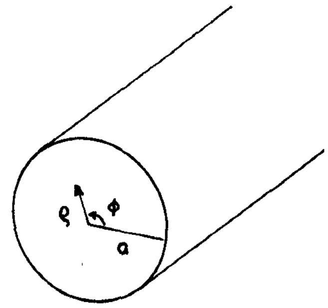

Fig. 3. The circular cylindrical waveguide.

$$\pi_1^{\pm} = J_n(h\rho)(A^{\pm}e^{jn\phi} + B^{\pm}e^{-jn\phi}). \tag{44}$$

Substituting this in (41), we obtain two possible sets of solutions. Either we have

$$ha\, J_n'(ha) \mp jn\, J_n(ha) = 0 \text{ and } B^{\pm} = 0 \tag{45}$$

or

$$ha\, J_n'(ha) \pm jn\, J_n(ha) = 0 \text{ and } A^{\pm} = 0. \tag{46}$$

The transcendental equation in (45) can also be written

$$ha\, J_{n\mp 1}(ha) = 0. \tag{47}$$

$h=0$ leads to a static mode, with π_1 satisfying the Laplace equation and because the fields vanish on the boundaries, the fields vanish everywhere and this case is noninteresting. Hence, only the zeros of J_{n+1} and J_{n-1} give the possible values for the eigenvalues: $h^n = p_{n\pm 1}/a$, if p is the zero for the Bessel function.

The general solutions with the eigenvalue p_m/a can be written as

$$\pi_{1m}^{\pm} = A_m^{\pm} J_{m-1}(p_m\rho/a)e^{\mp j(m-1)\phi} + B_m^{\pm} J_{m+1}(p_m\rho/a)e^{\pm j(m+1)\phi}. \tag{48}$$

Because the most general solution is a linear combination of these potentials, it is seen that in general, there is a fourfold degeneracy in the solution of the asymptotic fields of a circular cylindrical guide. For $m=0$ this is reduced to two, because the terms in (48) are multiples of each other as $J_{-1} = -J_1$. The electric field is written from (19) in the form

$$\boldsymbol{E}_{0m}^{\pm} = -j(\boldsymbol{I} \pm j\boldsymbol{u}\times\boldsymbol{I})\cdot\nabla\pi_{1m}^{\pm} \tag{49}$$

if $\boldsymbol{I}$ denotes the unit dyadic. Substituting (48) in (49) we have

$$\begin{aligned}\boldsymbol{E}_{0m}^{\pm} &= \pm j(p_m/a)(\boldsymbol{u}_\rho \pm j\boldsymbol{u}_\phi) \\ &\quad \cdot J_m(p_m\rho/a)e^{\pm j\phi}(A_m^{\pm}e^{\mp jm\phi} - B_m^{\pm}e^{\pm jm\phi}) \\ &= \pm j(p_m/a)(\boldsymbol{u}_x \pm j\boldsymbol{u}_y) \\ &\quad \cdot J_m(p_m\rho/a)(A_m^{\pm}e^{\mp jm\phi} - B_m^{\pm}e^{\pm jm\phi}).\end{aligned} \tag{50}$$

This field is obviously circularly polarized, as predicted. The basic asymptotic mode is obtained for the lowest eigenvalue h, which is $h=2.4/a$ for $m=0$. From (50) it is seen that as a linear combination of the self-dual solutions we may write for the lowest mode

$$E_{00}=A\boldsymbol{v}J_0(2.405\rho/a) \tag{51}$$

with a unit vector $\boldsymbol{v}$ transversal to $\boldsymbol{u}$. The field is then polarized in one constant direction everywhere in the waveguide, as was also obtained in [6].

For the general case, $m\neq 0$, the field can be written, instead of four self-dual fields, with four fields with constant linear polarizations

$$E_{0m}=\left[C_m^1\boldsymbol{u}_xe^{jm\phi}+C_m^2\boldsymbol{u}_xe^{-jm\phi}+C_m^3\boldsymbol{u}_ye^{jm\phi}+C_m^4\boldsymbol{u}_ye^{-jm\phi}\right]J_m\left(\frac{p_m\rho}{a}\right). \tag{52}$$

V. Attenuation in Waveguides

The attenuation is obtained from the next asymptotic equations. Writing the $n=2$ Helmholtz equations

$$(\nabla^2-2\beta_1)\begin{pmatrix}\pi_2\\ m_2\end{pmatrix}=2\beta_2\begin{pmatrix}\pi_1\\ m_1\end{pmatrix} \tag{53}$$

and the $n=1$ boundary conditions

$$\begin{pmatrix}\boldsymbol{n}\cdot\nabla & -\boldsymbol{t}\cdot\nabla\\ \boldsymbol{t}\cdot\nabla & \boldsymbol{n}\cdot\nabla\end{pmatrix}\begin{pmatrix}\pi_2\\ m_2\end{pmatrix}=2j\beta_1\begin{pmatrix}\eta/Z_s & 0\\ 0 & Z_s/\eta\end{pmatrix}\begin{pmatrix}\pi_1\\ m_1\end{pmatrix} \tag{54}$$

we are able to solve for the unknown coefficient β_2. The formula can be written in abstract form writing f for the potential pair (π, m), whence (53) reads $Lf_2=2\beta_2 f_1$ and (54) $Bf_2=2j\beta_1 Mf_1$. Here, L, B, and M are matrix operators. From (26) the following Green's formula can be derived:

$$(f,Lg)-(f,Bg)_b=(Lf,g)-(Bf,g)_b \tag{55}$$

if the inner products $(\cdot,\cdot)$, $(\cdot,\cdot)_b$ are integrals over the surface S and the boundary curve C, respectively. Thus, the operator pair L, B is self adjoint with respect to these inner products.

Substituting f_1^* for f and f_2 for g and noting that $Lf_1^*=0$ and $Bf_1^*=0$, we can solve for β_2

$$\beta_2=j\beta_1\frac{(f_1^*,Mf_1)_b}{(f_1^*,f_1)}=j\beta_1\frac{\oint\left[(\eta/Z_s)|\pi_1|^2+(Z_s/\eta)|m_1|^2\right]dC}{\int\left(|\pi_1|^2+|m_1|^2\right)dS}. \tag{56}$$

If the basic problem β_1, π_1, m_1 is solved, from (56) the next coefficient of β can be calculated.

The attenuation coefficient is defined as $\alpha=-\mathrm{Im}(\beta_2)/k^2$. It is seen at once that if Z_s is pure imaginary, we have $\alpha=0$, or the waveguide is lossless to the approximation. β_2, then, presents just a correction to the propagation coefficient. For the basic mode in the cylindrical guide (56) can be tested, and an expression equivalent to the equation (34) in [1] is obtained.

The potential coefficients π_2, m_2 are obtained by solving the deterministic problem (53), (54). The result is not unique, because a multiple of π_1, m_1 can be added to the solution. To obtain uniqueness, the solution f_2 can be required to be orthogonal to the solution f_1, or $(f_2, f_1)=0$ without any loss of generality, because the final solution to the potential problem (the asymptotic series) can be normalized.

A. Self-Dual Modes

For the self-dual modes, the formula (56) is simplified a little. From (39) we have $m_1^{\pm}=\mp j\pi_1^{\pm}$ and applying (42)

$$\beta_2^{\pm}=-j\beta_1\left(\frac{\eta}{Z_s}+\frac{Z_s}{\eta}\right)\frac{\oint|\pi_1^+|^2\,dC}{2\int|\pi_1^+|^2\,dS}. \tag{57}$$

For the self-dual modes the attenuation factors are the same. This is not valid for the general anisotropic boundary, however.

From (57) it is easy to see that the imaginary part of β_2 is minimum for $Z_s=\eta$, if the imaginary part of Z_s is zero. But if $\mathrm{Im}(Z_s)\neq 0$, then the minimum is attained at $\mathrm{Re}(Z_s)=0$. For good conductors, $\mathrm{Re}(Z_s)=\mathrm{Im}(Z_s)$ and the minimum is seen to exist at $\mathrm{Re}(Z_s)=\eta/\sqrt{2}$.

For the self-dual modes, some conclusions about the potentials can be made if the waveguide is lossless. Namely, if Z_s is imaginary, the operator jM is real and because L and B are real operators, from (53), (54) we see that π_2^*, m_2^* is a solution with the same β_2 as π_2, m_2. In this case, there is the following dependence between the self-dual solutions:

$$\pi_2^-=(\pi_2^+)^* \qquad m_2^-=(m_2^+)^*. \tag{58}$$

B. Anisotropic Boundary

We consider boundary conditions more general than (4), i.e., conditions of the form

$$\boldsymbol{n}\times\boldsymbol{E}=\boldsymbol{Z}_s\cdot\boldsymbol{H} \tag{59}$$

where $\boldsymbol{Z}_s$ is the surface impedance dyadic. It is a two-dimensional dyadic, i.e., orthogonal to $\boldsymbol{n}$: $\boldsymbol{n}\cdot\boldsymbol{Z}_s=\boldsymbol{Z}_s\cdot\boldsymbol{n}=0$ and can be written in the form

$$\boldsymbol{Z}_s=Z_{tt}\boldsymbol{tt}+Z_{tz}\boldsymbol{tu}+Z_{zt}\boldsymbol{ut}+Z_{zz}\boldsymbol{uu}. \tag{60}$$

Denoting $\det\boldsymbol{Z}_s=Z_{tt}Z_{zz}-Z_{tz}Z_{zt}$, we may write (59) also as follows:

$$\begin{pmatrix}E_t\\ \eta H_t\end{pmatrix}=-\frac{1}{Z_{tt}}\begin{pmatrix}Z_{zt} & (\det\boldsymbol{Z}_s)/\eta\\ -\eta & Z_{tz}\end{pmatrix}\begin{pmatrix}E_z\\ \eta H_z\end{pmatrix}. \tag{61}$$

For Hertzian potentials this takes on the form

$$\begin{pmatrix}k\boldsymbol{n}\cdot\nabla & -\beta\boldsymbol{t}\cdot\nabla\\ \beta\boldsymbol{t}\cdot\nabla & k\boldsymbol{n}\cdot\nabla\end{pmatrix}\begin{pmatrix}\pi\\ m\end{pmatrix}=-\frac{jk_c^2}{Z_{tt}}\begin{pmatrix}\eta & -Z_{tz}\\ Z_{zt} & (\det\boldsymbol{Z}_s)/\eta\end{pmatrix}\begin{pmatrix}\pi\\ m\end{pmatrix} \tag{62}$$

which obviously generalizes the condition (5).

The basic asymptotic quantities, β_1, π_1, m_1, are not affected by the anisotropy, but the formula for β_2, (56), is changed because the operator M is changed. Hence, the

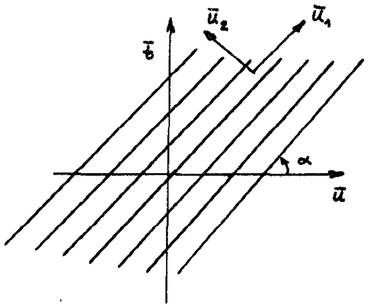

Fig. 4. Helical corrugations on the waveguide surface.

more general form of (56) can be written as follows:

$$\beta_2=\frac{-j\beta_1}{Z_{tt}}\frac{\oint\left(\eta|\pi_1|^2-Z_{tz}\pi_1^*m_1+Z_{zt}m_1^*\pi_1+\frac{1}{\eta}\det Z_s|m_1|^2\right)dC}{\int\left(|\pi_1|^2+|m_1|^2\right)dS}. \tag{63}$$

For the self-dual modes satisfying (39) and (42) we have instead of (57)

$$\beta_2^{\pm}=\frac{-j\beta_1}{Z_{tt}}\left[\eta+\frac{\det \boldsymbol{Z}_s}{\eta}\pm j(Z_{tz}+Z_{zt})\right]\frac{\int|\pi_1^+|^2\,dC}{2\int|\pi_1^+|^2\,dS}. \tag{64}$$

From this we see that unless $Z_{tz}+Z_{zt}=0$, the two self-dual modes have $\beta_2^+\neq\beta_2^-$. For a reciprocal surface material we always have a symmetric dyadic $\boldsymbol{Z}_s$, whence only for a diagonal $\boldsymbol{Z}_s$ (i.e., with eigenvectors along $\boldsymbol{u}$ and $\boldsymbol{t}$), the propagation coefficients have the same value. This is satisfied for some boundary materials, e.g., for a corrugated surface material. More general surface impedance dyadics can be realized with a layer of magnetized ferrite material [7].

From (63) we can speculate how to obtain a small attenuation. Obviously, the parameter Z_{tt} should be large with respect to η. Denoting $\eta/Z_{tt}=p$, a small parameter, the impedance parameters Z_{tz} and Z_{zt} should be of the order 0(1) and Z_{zz} of the order $0(p)$, in order that the attenuation factor be $0(p)$. Interpreted from the equation (59), this implies the two conditions: $\boldsymbol{t}\cdot\boldsymbol{H}$ and $\boldsymbol{t}\cdot\boldsymbol{E}$ must both be of the order $0(p)$, or they must be small. This is well approximated by the corrugated surface.

C. Corrugated Waveguide

We consider the general corrugated waveguide with corrugations directed helically with an angle α to the axis $\boldsymbol{u}$. (See Fig. 4.) In the local frame, we can write

$$Z_s=Z_1\boldsymbol{u}_1\boldsymbol{u}_1+Z_2\boldsymbol{u}_2\boldsymbol{u}_2$$
$$\begin{pmatrix}\boldsymbol{u}_1\\ \boldsymbol{u}_2\end{pmatrix}=\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{t}\end{pmatrix}. \tag{65}$$

Because $\det \boldsymbol{Z}_s=Z_1Z_2$, (64) can be written in the form

$$\beta_2^{\pm}=-j\beta_1\frac{\eta^2+Z_1Z_2\pm j\eta(Z_2-Z_1)\sin 2\alpha}{\eta(Z_1\sin^2\alpha+Z_2\cos^2\alpha)}\cdot\frac{\oint|\pi_1^+|^2\,dC}{2\int|\pi_1^+|^2\,dS}. \tag{66}$$

If $Z_1=Z_2$ this reduces to (57). If Z_1 and Z_2 are real, e.g., lossy tuned corrugations, it is not difficult to find out that if $Z_1>Z_2$, the smallest attenuation is obtained for $\alpha=\pi/2$, which corresponds to the conventional transversal corrugations. The maximum attenuation is obtained for $\alpha=0$, or longitudinal corrugations.

For transverse corrugations, (66) reads

$$\beta_2^{\pm}=-j\beta_1\left(\frac{\eta}{Z_{tt}}+\frac{Z_{zz}}{\eta}\right)\frac{\oint|\pi_1^+|^2\,dC}{2\int|\pi_1^+|^2\,dS} \tag{67}$$

and it is seen that to have a small attenuation, the corrugations must be tuned to give Z_{tt} the largest possible value.

VI. Conclusion

Asymptotic analysis was applied to the problem of wave propagation in large waveguides with impedance boundaries. The purpose was to expand the classical waveguide analysis in this direction and no claim of completeness in solving problems arising in practical oversized waveguides was made. To proceed in this field, one has to take into account the nondeterministic defects of the waveguide boundary, for example.

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